# Autonomous averaging and numerics

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Strasburg PDE seminar
18/10/2022

# Outline

1 The averaging procedure and numerical accuracy
- Order reduction for the stiff Hénon-Heiles model
- Overcoming the order reduction through averaging

2 The averaging procedure and geometry
- Geometric considerations
- Overcoming the asymptote

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# A general model

We are concerned with models of the form

$$\partial_t y^\varepsilon = \frac{1}{\varepsilon} A y^\varepsilon + f(y^\varepsilon), \qquad y^\varepsilon(0) = y_0 \in X$$

## Assumptions

- The space $(X, |\cdot|)$ is a Banach
- The operator $A$ generates a $2\pi$-periodic group $\theta \mapsto e^{\theta A}$
- The vector field $f$ is *analytic* and bounded in an open subset $\mathcal{K}$
- For all $\varepsilon$ small enough, the solution stays in $\mathcal{K}$ for $t \in [0, 1]$

Some examples:

- Non-linear Schrödinger
- Non-relativistic Klein-Gordon
- Vlasov equation with stiff magnetic field
- Stiff Hénon-Heiles model

# The stiff Hénon-Heiles model – Equations

Consider the Hénon-Heiles model with a stiff direction

$$\begin{cases} \dot{q}_1 = \frac{1}{\varepsilon} p_1 \\ \dot{q}_2 = p_2 \\ \dot{p}_1 = -\frac{1}{\varepsilon} q_1 - 2 q_1 q_2 \\ \dot{p}_2 = -q_2 - {q_1}^2 + {q_2}^2 \end{cases}$$

## Property – Error of standard schemes

For an usual numerical scheme of order $q$, the error is of the form

$$|y^\varepsilon(t_n) - y_n| \leq C \, \Delta t^q \left\| {\partial_t}^{q+1} y^\varepsilon \right\|_{L^1}$$

where $C$ depends on the scheme.

- Consider schemes that take into account the highly-oscillatory nature of the problem.

# The stiff Hénon-Heiles model – Simulations

# Filtering the oscillations

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- Consider schemes that take into account the highly-oscillatory nature of the problem.

To that effect, we *filter out* the main oscillations by setting

$$\boxed{u^\varepsilon(t) = e^{-tA/\varepsilon} y^\varepsilon(t), \qquad \text{i.e.} \qquad \partial_t u^\varepsilon(t) = g_{t/\varepsilon}(u^\varepsilon(t))}$$

with $g_\theta = e^{-\theta A} f \circ e^{\theta A}$.

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# The stiff Hénon-Heiles model – Filtering

We *filter* the oscillations with the symplectic change of variable

$$\begin{cases} u_1(t) = \cos(t/\varepsilon) q_1(t) - \sin(t/\varepsilon) p_1(t) \\ u_2(t) = q_2(t) \\ u_3(t) = \sin(t/\varepsilon) q_1(t) + \cos(t/\varepsilon) p_1(t) \\ u_4(t) = p_2(t) \end{cases}$$

yielding $\partial_t u^\varepsilon(t) = g_{t/\varepsilon}\big(u^\varepsilon(t)\big)$ with

$$g_\theta(u) = \left| \begin{array}{l} 2u_2 \sin(\theta)\big(u_1 \cos(\theta) + u_3 \sin(\theta)\big) \\ u_4 \\ -2u_2 \cos(\theta)\big(u_1 \cos(\theta) + u_3 \sin(\theta)\big) \\ -u_2 - \big(u_1 \cos(\theta) + u_3 \sin(\theta)\big)^2 + u_2^2 \end{array} \right.$$

This new equation in $u^\varepsilon$ is better posed!

# Integral numerical schemes

On this new, non-autonomous problem, we may use *integral* schemes

$$u_{\ell+1} = u_n + \int_{t_n}^{t_{\ell+1}} g_{t/\varepsilon}(u_n)\mathrm{d}t \tag{RK1}$$

$$\begin{cases} u_{\ell+1/2} = u_\ell + \displaystyle\int_{t_n}^{t_n+\Delta t/2} g_{t/\varepsilon}(u_\ell)\mathrm{d}t \\ u_{\ell+1} = u_\ell + \displaystyle\int_{t_n}^{t_{\ell+1}} g_{t/\varepsilon}(u_{\ell+1/2})\mathrm{d}t \end{cases} \tag{RK2}$$

## Property – Error of integral schemes

The error of such a scheme of order $q$ is of the form

$$|u^\varepsilon(t_\ell) - u_\ell| \leq C\,\Delta t^q \left\|\partial_t{}^q u^\varepsilon\right\|_{L^1}$$

where $C$ depends on the scheme.

# Order reduction

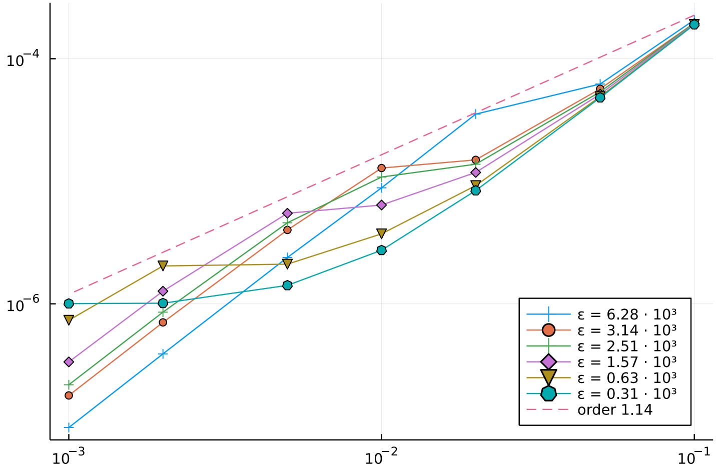


Figure: Numerical error on the Hénon-Heiles model for $t \in [0, 1]$ when simulating with the RK2 scheme.

# New notions of convergence

$$
\begin{tikzcd}
u^{\varepsilon_0}(t) \arrow[rr, "\varepsilon \to 0"] & {} & u^0(t) \\
(u_\ell^{\varepsilon_0}) \arrow[u, "\mathcal{O}(\Delta t^q)"] \arrow[rr, "\varepsilon \to 0"'] & {} \arrow[u, dashed, "\mathcal{O}(\Delta t^q)", "\mathrm{UA}"'] & (u_\ell^0) \arrow[u, "\mathcal{O}(\Delta t^q)", "\mathrm{AP}"']
\end{tikzcd}
$$

**Definition – Order of convergence**

A method of order $q$ is said to be *uniformly accurate* (UA) if its uniform order of convergence is not degraded, i.e. if

$$\sup_{0<\varepsilon\leq\varepsilon_0} \max_{0\leq\ell\leq N} |u^\varepsilon(t_\ell) - u_\ell^\varepsilon| = \mathcal{O}(\Delta t^q).$$

We say a method is *asymptotic preserving* (AP) if

$$\lim_{\varepsilon\to 0^+} \max_{0\leq\ell\leq N} |u^\varepsilon(t_\ell) - u_\ell^\varepsilon| = \mathcal{O}(\Delta t^q).$$

These notions may depend on the initial conditions (e.g. near-equilibrium).

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# Context and assumptions

$$\partial_t u^\varepsilon(t) = g_{t/\varepsilon}\big(u^\varepsilon(t)\big), \qquad u^\varepsilon(0) = u_0 \in \mathcal{X}_0 \subset X$$

## Assumptions on the vector field

- $(X, |\cdot|)$ is a Banach
- The map $(\theta, u) \mapsto g_\theta(u)$ is $2\pi$-periodic w.r.t. $\theta$
- The problem is well-posed up to $t = 1$ for all $\varepsilon$
- The solution stays in $\mathcal{K} \subset X$

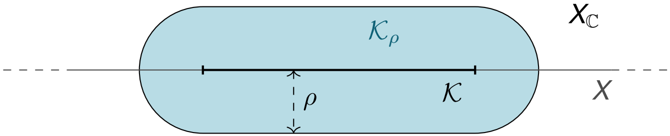
The vector field may stem from the *filtering* $u^\varepsilon(t) = e^{-\frac{t}{\varepsilon}A} y^\varepsilon(t)$ of an autonomous problem

$$\dot{y}^\varepsilon = \frac{1}{\varepsilon} A y^\varepsilon + f(y^\varepsilon) \quad \Leftrightarrow \quad \partial_t u^\varepsilon(t) = e^{-\frac{t}{\varepsilon}A} f\big(e^{\frac{t}{\varepsilon}A} u^\varepsilon(t)\big).$$

# Analyticity and complex expansions

We introduce the complex extensions of $\mathcal{K}$ as

$$\mathcal{K}_\rho := \{u + \tilde{u}, \quad (u, \tilde{u}) \in \mathcal{K} \times X_{\mathbb{C}}, |\tilde{u}|_{\mathbb{C}} \leq \rho\}\,.$$




## Assumption – Analyticity

There exists some $R > 0$ such that the vector field $(\theta, u) \mapsto g_\theta(u)$ is $u$-analytic in $\mathcal{K}$, of radius everywhere greater than $2R$. Furthermore,

$$\sup_{(\theta,u)\in\mathbb{T}\times\mathcal{K}_{2R}} |g_\theta(u)| =: \|g\|_{2R} \leq M,$$

where $g_\theta(u + \widetilde{u})$ is defined from a Taylor series around $u$.

# Average behaviour

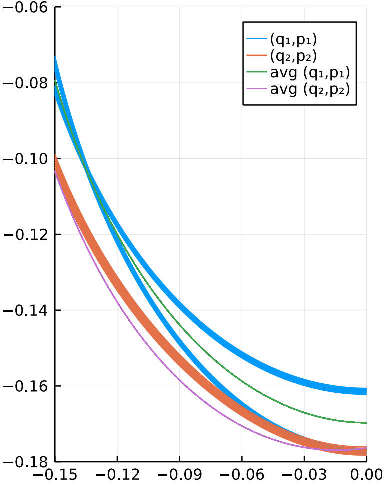



Figure: Exact and average dynamics for $\varepsilon = 0.5$ up to $t = 50$.

The main dynamics are dictated by the *non-stiff* problem

$$\partial_t \overline{u} = \langle g \rangle (\overline{u}),$$

with the average vector field

$$\boxed{\langle g \rangle = \frac{1}{2\pi} \int_0^{2\pi} g_\theta \mathrm{d}\theta.}$$

# The averaging ansatz

We intend to decompose the solution $u^\varepsilon$ as

$$u^\varepsilon(t) = \Phi^\varepsilon_{t/\varepsilon} \circ \Psi^\varepsilon_t \circ \left(\Phi^\varepsilon_0\right)^{-1}(u_0)$$

with $(\theta, u) \mapsto \Phi^\varepsilon_\theta(u)$ a $2\pi$-periodic change of variable and $(t, u) \mapsto \Psi^\varepsilon_t(u)$ the flow of an autonomous equation.

$$\Phi^\varepsilon_\theta(u) = u + \mathcal{O}(\varepsilon) \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\Psi^\varepsilon_t(u) = G^\varepsilon \circ \Psi^\varepsilon_t(u).$$

In general, $\Phi^\varepsilon$ and $G^\varepsilon$ can only be expressed as *diverging* formal series!

There are generally two types of averaging

| | |
|---|---|
| Standard | The simplest choice, $\langle \Phi^\varepsilon \rangle = \mathrm{id}$. |
| **Stroboscopic** | A less direct but more geometric choice $\Phi^\varepsilon_0 = \mathrm{id}$. |

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# Some literature

What I will talk about can be found in

- Chartier, Lemou, Méhats, and Vilmart (FoCM 2020)
- Chartier, Lemou, Méhats, and Trémant (in preparation)

My works extend this framework to other contexts

- Relaxation problems with Chartier, Lemou (Math. Comp. 2022)
- Multi-frequency problems with an added relaxation part, with Bidégaray-Fesquet, Jourdana (in preparation)

Close to other methods of asymptotic/two-scale expansion

- Homogeneisation
- Chapman-Enskog expansion
- Non-linear geometric optics

# The homological equation

After injecting the ansatz in the equation on $u^\varepsilon$, we obtain

$$\partial_\theta \Phi_\theta^\varepsilon(u) = \varepsilon\Big(g_\theta \circ \Phi_\theta^\varepsilon(u) - \partial_u \Phi_\theta^\varepsilon(u) \cdot F^\varepsilon(u)\Big).$$

Taking the average of this equation yields

$$G^\varepsilon = \big\langle \partial_u \Phi^\varepsilon(u) \big\rangle^{-1} \langle g \circ \Phi^\varepsilon \rangle(u).$$

This may therefore be written

$$\partial_\theta \Phi^\varepsilon = \varepsilon \Lambda\big(\Phi^\varepsilon\big)$$

The study of averaging may now focus on the properties of this operator $\Lambda$ !

This is mainly where this *closed form* approach differs from usual multi-scale expansions.

# Well-posedness of averaging

The homological equation is "solved" iteratively with the relation

$$\partial_\theta \Phi_\theta^{[n+1]} = \varepsilon \Lambda(\Phi^{[n]})_\theta \quad \text{and} \quad \Phi^{[0]} = \mathrm{id}.$$

with either closure condition $\langle \Phi^{[n]} \rangle = \mathrm{id}$ or $\Phi_0^{[n]} = \mathrm{id}$. Denote $\delta^{[n]}$ the defect,

$$\delta_\theta^{[n]} = \frac{1}{\varepsilon} \partial_\theta \Phi_\theta^{[n]} - \Lambda(\Phi^{[n]})_\theta = \Lambda(\Phi^{[n-1]})_\theta - \Lambda(\Phi^{[n]})_\theta$$

## Properties of the averaging procedure

For all $n \in \mathbb{N}$, the averaging procedure is well defined for $0 < \varepsilon \le \varepsilon_n$. Specifically, with $(n+1)\varepsilon_n = M/(16R)$, then

$$\|\Phi^{[n]} - \mathrm{id}\,\|_R \le \frac{\varepsilon}{2\varepsilon_n} R, \qquad \|G^{[n]}\|_R \le 2M, \qquad \|\delta^{[n]}\| \le 2M \left(\frac{\varepsilon}{\varepsilon_n}\right)^n$$

with in addition $\langle \delta^{[n]} \rangle = 0$. Their derivatives w.r.t. $\theta$ are of the same size, i.e. respectively $\mathcal{O}(\varepsilon)$, $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon^n)$.

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# Asymptotic approximation of the solution

Considering a *micro-macro* decomposition,

$$u^\varepsilon(t) = \Phi_{t/\varepsilon}^{[n]}\big(v(t)\big) + w(t)$$

with $v(t) = \Psi_t^{[n]}(u_0)$, then

$$\partial_t w = g_{t/\varepsilon}\big(\Phi_{t/\varepsilon}^{[n]}(v) + w\big) - g_{t/\varepsilon}\big(\Phi_{t/\varepsilon}^{[n]}(v)\big) - \delta_{t/\varepsilon}^{[n]}(v)$$

- This is a quasi-linear equation with a source term !

A direct application of Gronwall's lemma yields

$$\forall t \in [0,1], \quad \left|u^\varepsilon(t) - \Phi_{t/\varepsilon}^{[n]} \circ \Psi_t^{[n]}(u_0)\right| \leq C\left(\frac{\varepsilon}{\varepsilon_n}\right)^{n+1},$$

**Note :** By choosing $n$ in function of $\varepsilon$, we may obtain an error bound of the form

$$\forall t \in [0,1], \quad \left|u^\varepsilon(t) - \Phi_{t/\varepsilon}^{[n(\varepsilon)]} \circ \Psi_t^{[n(\varepsilon)]}(u_0)\right| \leq Ce^{-\nu/\varepsilon}$$

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# Micro-macro decomposition

Considering a *micro-macro* decomposition,

$$\begin{cases} \partial_t v = G^{[n]}(v) \\ \partial_t w = g_{t/\varepsilon}\big(\Phi^{[n]}_{t/\varepsilon}(v) + w\big) - g_{t/\varepsilon}\big(\Phi^{[n]}_{t/\varepsilon}(v)\big) - \delta^{[n]}_{t/\varepsilon}(v) \end{cases}$$

## Proposition – Uniform accuracy on the micro-macro problem

This problem is non-stiff up to its $(n+1)$-th derivative and can therefore be solved with uniform accuracy up to order $n$ for standard schemes and order $n+1$ with integral schemes. In other words, we may obtain

$$|v(t_\ell) - v_\ell|, |w(t_\ell) - w_\ell| \leq C\Delta t^{n+1}$$

with $C$ independent of $\varepsilon$.

We finally recover an approximation of $u^\varepsilon$ by setting

$$u^\varepsilon_\ell = \Phi^{[n]}_{t_\ell/\varepsilon}(v_\ell) + w_\ell$$

# Uniform accuracy of the micro-macro decomposition

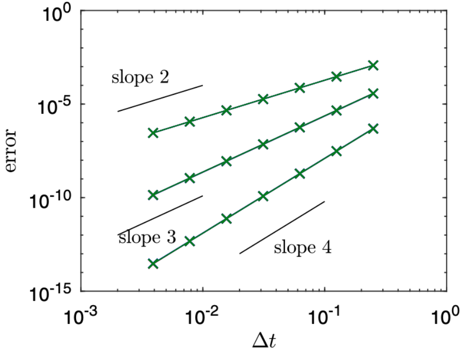


Figure: Error as a function of $\Delta t$ for $\varepsilon = 2^{-k}$ for $k \in \{0, 1, \ldots, 9\}$ with the micro-macro decomposition and integral numerical schemes.

From: Philippe Chartier, Mohammed Lemou, Florian Méhats, and Gilles Vilmart (Feb. 2020). “A New Class of Uniformly Accurate Numerical Schemes for Highly Oscillatory Evolution Equations”. In: *Foundations of Computational Mathematics* 20.1. ISSN: 1615-3375, 1615-3383

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# The Hamiltonian case

We consider Hamiltonian systems in finite dimensions,

$$X = \mathbb{R}^{2d}.$$

## Definition – Hamiltonian structure

A vector field $(\theta, u) \mapsto g_\theta(u)$ is said to be *Hamiltonian* if there exists $(\theta, u) \mapsto H_\theta(u) \in \mathbb{R}$ such that

$$g_\theta(u) = J^{-1}\, \nabla_u H_\theta(u) \quad \text{where} \quad J = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}.$$

A mapping $u \mapsto \phi(u)$ is *symplectic* if it preserves the structure, i.e. if

$$\partial_u \phi(u)\, J^{-1} \big(\partial_u \phi(u)\big)^T = J^{-1}$$

**Note:** If $y$ follows a Hamiltonian vector field and $\phi$ is symplectic, then $u = \phi(y)$ also follows a Hamiltonian vector field.

# The stiff Hénon-Heiles model – Hamiltonian

In the case of the stiff Hamiltonian model,

$$H(q_1, q_2, p_1, p_2) = \underbrace{\frac{1}{2\varepsilon}\left({p_1}^2 + {q_1}^2\right)}_{\frac{1}{\varepsilon}H_0} + \underbrace{\frac{1}{2}\left({p_2}^2 + {q_2}^2\right) + {q_1}^2 q_2 - \frac{1}{3}{q_2}^3}_{H_1}$$

such that the system is written

$$\dot{q} = \nabla_p H(p, q), \qquad \dot{p} = -\nabla_q H(p, q)$$

After filtering with the change of variable $u^\varepsilon(t) = e^{-\frac{t}{\varepsilon}J^{-1}\nabla H_0} y^\varepsilon(t)$, the so-called *filtered* Hamiltonian is

$$H_\theta = H_1 \circ e^{\theta J^{-1}\nabla H_0}$$

$$H_\theta(u) = \frac{1}{2}\left({u_2}^2 + {u_4}^2\right) + \left(u_1\cos(\theta) + u_3\sin(\theta)\right)^2 u_2 - \frac{1}{3}{u_2}^3$$

# Flow and Hamiltonian problem

Considering a (possibly) time-dependent vector field $(t, u) \mapsto f_t(u)$, the canonical associated flow is defined

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t = f_t \circ \varphi_t, \qquad \varphi_0 = \mathrm{id},$$

such that

$$y(t) = \varphi_t(y_0) \quad \Leftrightarrow \quad \begin{cases} \dot{y} = f_t(y), \\ y(0) = y_0. \end{cases}$$

## Properties of the flow

- The flow $(t, u) \mapsto \varphi_t(u)$ of an ODE is symplectic *if and only if* the associated vector field is Hamiltonian, i.e. $f_t = J^{-1}\nabla H_t$.
- In the *autonomous Hamiltonian* case $H_t = H$, the Hamiltonian is preserved by the flow.

As such, the previous filtering $e^{-\theta J^{-1}\nabla H_0}$ is naturally symplectic.

# Energy drift

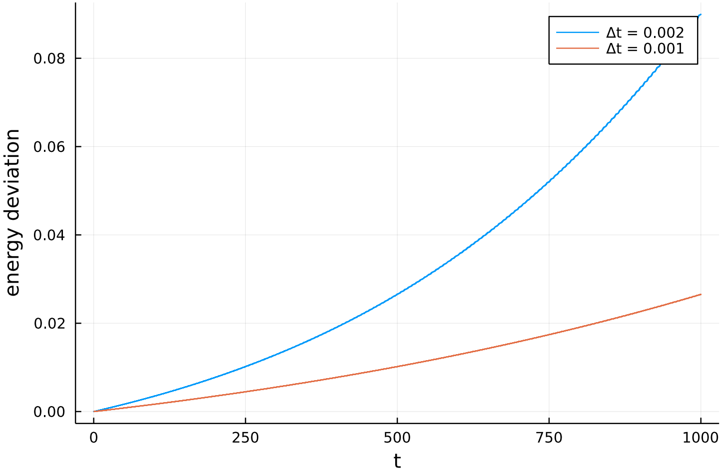


Figure: Energy deviation for the RK1 scheme with $\varepsilon = \frac{\pi}{100}$.

# The midpoint scheme – Energy

$$u_{\ell+1} = u_\ell + \int_{t_\ell}^{t_{\ell+1}} g_\theta\left(\frac{1}{2}(u_\ell + u_{\ell+1})\right) \mathrm{d}\theta$$

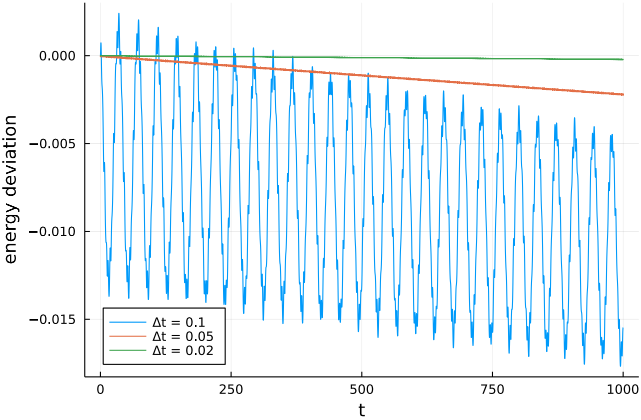


Figure: Energy deviation for the midpoint scheme with $\varepsilon = \pi/100$.

# The midpoint scheme – Accuracy

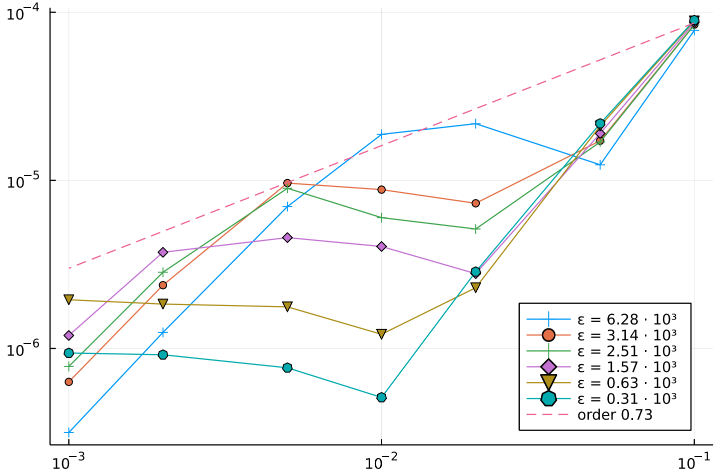


Figure: Numerical error on the Hénon-Heiles model for $t \in [0, 1]$ when simulating with the integral midpoint scheme.

# Asymptotic geometric preservation

As it turns out, given maps $\Phi^{[n]}$ and $G^{[n]} = \mathcal{O}(1)$, the quality of approximation

$$\forall t \in [0,1], \quad \left| u^\varepsilon(t) - \Phi^{[n]}_{t/\varepsilon} \circ \Psi^{[n]}_t(u_0) \right| = \mathcal{O}(\varepsilon^{n+1}),$$

with $\frac{\mathrm{d}}{\mathrm{d}t}\Psi^{[n]}_t = G^{[n]}\big(\Psi^{[n]}_t\big)$, is enough to obtain the following geometric result.

## Theorem – Geometric conservation

If the flow generated by $(t,u) \mapsto g_{t/\varepsilon}(u)$ presents a geometric property such as if

1. it is volume-preserving
2. it preserves $I : \mathbb{R}^{2d} \to \mathbb{R}$
3. it is symplectic
4. it is $B$-symplectic

then for $\varepsilon$ small, the flow $(t,u) \mapsto \Psi^{[n]}_t(u)$ presents the same property up to $\mathcal{O}(\varepsilon^{n+1})$.

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# Pullback approach

This approach performs a sort of “high-order filtering” by setting

$$\boxed{\Phi^{[n]}_{t/\varepsilon}\big(v(t)\big) = u^\varepsilon(t).}$$

- If $\Phi^{[n]}$ is symplectic, then $v$ satisfies a Hamiltonian equation.

This “pulled-back” or filtered variable satisfies

$$\partial_t v(t) = \big(\partial_u \Phi^{[n]}_{t/\varepsilon}(v)\big)^{-1}\left(g_\theta \circ \Phi^{[n]}_{t/\varepsilon}(v) - \frac{1}{\varepsilon}\partial_\theta \Phi^{[n]}_{t/\varepsilon}(v)\right)$$

How do we ensure the symplecticity of $\Phi^{[n]}$?

For the well-posedness of the problem, we remark the identity

$$\partial_t v(t) = G^{[n]}(v) - \big(\partial_u \Phi^{[n]}_{t/\varepsilon}(v)\big)^{-1}\delta^{[n]}_{t/\varepsilon}(v)$$

# Pullback approach

This approach performs a sort of “high-order filtering” by setting

$$\boxed{\Phi^{[n]}_{t/\varepsilon}\big(v(t)\big) = u^\varepsilon(t).}$$

- If $\Phi^{[n]}$ is symplectic, then $v$ satisfies a Hamiltonian equation.

This “pulled-back” or filtered variable satisfies

$$\partial_t v(t) = \left(\partial_u \Phi^{[n]}_{t/\varepsilon}(v)\right)^{-1}\left(g_\theta \circ \Phi^{[n]}_{t/\varepsilon}(v) - \frac{1}{\varepsilon}\partial_\theta \Phi^{[n]}_{t/\varepsilon}(v)\right)$$

How do we ensure the symplecticity of $\Phi^{[n]}$?

For the well-posedness of the problem, we remark the identity

$$\partial_t v(t) = G^{[n]}(v) - \left(\partial_u \Phi^{[n]}_{t/\varepsilon}(v)\right)^{-1}\delta^{[n]}_{t/\varepsilon}(v)$$

# Correcting the change of variable

Consider the map at fixed $\theta$, and write it as

$$\Phi_\theta^{[1]} = \mathrm{id} + \varepsilon\,\omega_\theta^{[1]} \quad \text{“} = \exp\left(\varepsilon\,\omega_\theta^{[1]}\right) + \mathcal{O}(\varepsilon^2)\text{”}$$

with $\omega_\theta^{[1]}$ Hamiltonian! We may therefore introduce a formal variable $s$ and modify $\Phi_\theta^{[1]}$ up to $\mathcal{O}(\varepsilon^2)$ with

$$\Phi_\theta^{[1]} = \mathcal{U}_{\theta,s}^{[1]}\Big|_{s=1} \qquad \text{where} \quad \partial_s\,\mathcal{U}_{\theta,s}^{[1]} = \varepsilon\,\omega_\theta^{[1]} \circ \mathcal{U}_{\theta,s}^{[1]}, \quad \mathcal{U}_{\theta,0}^{[1]} = \mathrm{id}\,.$$

This may be integrated using a *midpoint method,*

$$\boxed{\Phi_\theta^{[1]} = \mathrm{id} + \varepsilon\,\omega_\theta^{[1]} \circ \left(\frac{1}{2}(\mathrm{id} + \Phi_\theta^{[1]})\right).}$$

For order 2, we identify $\omega_\theta^{[2]}$ such that

$$\Phi_\theta^{[2]} = \mathrm{id} + \varepsilon\,\omega_\theta^{[2]} + \frac{\varepsilon^2}{2}\partial_u\omega_\theta^{[2]} \cdot \omega_\theta^{[2]} + \mathcal{O}(\varepsilon^3)$$

and the same reasoning holds.

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## Computing the new pullback equation

The midpoint scheme may be differentiated to obtain

$$\frac{1}{\varepsilon}\partial_\theta \Phi_\theta^{[1]} = \partial_\theta \omega_\theta^{[1]}\big(\Phi_\theta^{[1/2]}\big) + \partial_u \omega_\theta^{[1]}\big(\Phi_\theta^{[1/2]}\big) \cdot \frac{\varepsilon}{2}\left[\frac{1}{\varepsilon}\partial_\theta \Phi_\theta^{[1]}\right]$$

$$\big(\partial_u \Phi_\theta^{[1]}\big)^{-1} = \mathrm{id} - \varepsilon\, \partial_u \omega_\theta^{[1]}\big(\Phi_\theta^{[1/2]}\big) \cdot \frac{1}{2}\left(\mathrm{id} + \big(\partial_u \Phi_\theta^{[1]}\big)^{-1}\right)$$

where we denoted $\Phi_\theta^{[1/2]} = \frac{1}{2}\big(\mathrm{id} + \Phi_\theta^{[1]}\big)$.

The pullback equation is obtained from its definition

$$\partial_t v(t) = \left(\partial_u \Phi_{t/\varepsilon}^{[1]}(v)\right)^{-1}\left(g_\theta \circ \Phi_{t/\varepsilon}^{[1]}(v) - \frac{1}{\varepsilon}\partial_\theta \Phi_{t/\varepsilon}^{[1]}(v)\right)$$

This new problem improves numerical accuracy and preserves geometric structures!

# Uniform accuracy of the pullback method

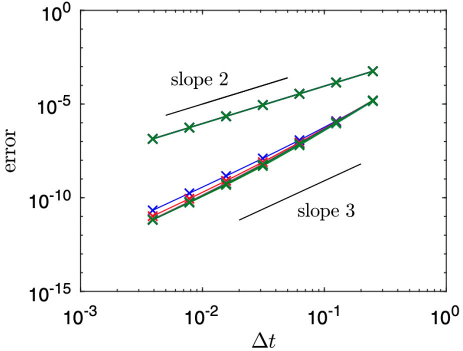
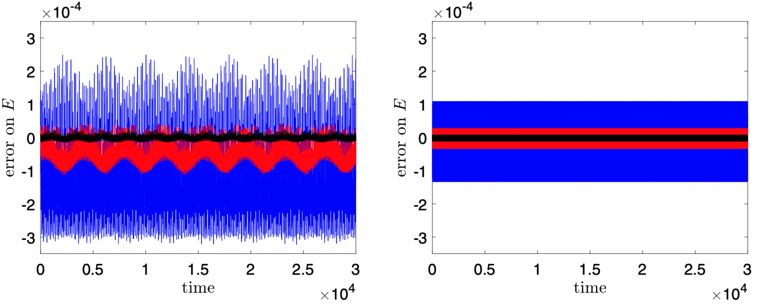



Figure: Error as a function of $\Delta t$ for $\varepsilon = 2^{-k}$ for $k \in \{0, 1, \ldots, 9\}$ with the pullback problem and integral numerical schemes.

From: Philippe Chartier, Mohammed Lemou, Florian Méhats, and Gilles Vilmart (Feb. 2020). "A New Class of Uniformly Accurate Numerical Schemes for Highly Oscillatory Evolution Equations". In: *Foundations of Computational Mathematics* 20.1. ISSN: 1615-3375, 1615-3383

# Evolution of energy with the pullback method




**(a)** $\varepsilon = 1$. **(b)** $\varepsilon = 0.001$.

Figure: Error evolution on the Hamiltonian for the pulled-back midpoint method (blue: $\Delta t = 0.2$, red: $\Delta t = 0.1$) and for a method of order 3 (black).

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# Standard averaging of an autonomous problem

When applying this method on the filtration of an autonomous problem, i.e. remember

$$\dot{y}^\varepsilon = \frac{1}{\varepsilon} A y^\varepsilon + f(y^\varepsilon) \quad \Leftrightarrow \quad \partial_t u^\varepsilon(t) = e^{-\frac{t}{\varepsilon}A} f\big(e^{\frac{t}{\varepsilon}A} u^\varepsilon(t)\big),$$

then the maps resulting of *standard averaging* satisfy

$$\Phi_\theta^{[n]} = \Phi_0^{[n]} \circ e^{\theta A}, \qquad \big[G^{[n]}, A\big] = 0, \qquad \delta_\theta^{[n]} = \delta_0^{[n]} \circ e^{\theta A}.$$

- Setting $\widetilde{v}(t) = e^{tA/\varepsilon} v(t)$ and $\widetilde{w}(t) = e^{tA/\varepsilon} w(t)$, the micro-macro problem becomes

$$\begin{cases} \partial_t \widetilde{v} = \frac{1}{\varepsilon} A \widetilde{v} + G^{[n]}(\widetilde{v}) \\ \partial_t \widetilde{w} = \frac{1}{\varepsilon} A \widetilde{w} + f\big(\Phi_0^{[n]}(\widetilde{v}) + \widetilde{w}\big) - f\big(\Phi_0^{[n]}(\widetilde{v})\big) - \delta_0^{[n]}(\widetilde{v}) \end{cases}$$

and the first equation can be solved using a Lie splitting with no error.

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and the first equation can be solved using a Lie splitting with no error.

# Principle of normal forms

James Murdock (2006). *Normal forms and unfoldings for local dynamical systems*. Springer Science & Business Media

The idea is to find a change of variable $\tau^{[n]}$ such that

$$A + \varepsilon f = \left(\partial_u \tau^{[n]}\right)^{-1} \left(A + \varepsilon f^{[n]} + \varepsilon^{n+1} R^{[n]}\right) \circ \tau^{[n]}$$

with $[A, f^{[n]}] = 0$. Usually, $\varepsilon$ represents the distance from a rest point.

**Note :** In the literature, “normal forms” are usually used for theoretical results and “non-linear changes of variable” are for numerics.

Many methods exist to construct this $\tau^{[n]}$ depending on the context and the goal.

This $\tau^{[n]}$ acts in the same way as $\left(\Phi_0^{[n]}\right)^{-1}$.

# Classification of normal forms

Murdock basically distinguishes 3 methods to compute normal forms

1. Direct method

$$\tau^{[n]} = \mathrm{id} + \varepsilon\tau_1 + \cdots + \varepsilon^n\tau_n$$

➡ Akin to *standard* averaging.

2. Deprit's method

$$\tau^{[n]} = \tau_s^{[n]}\Big|_{s=1} \qquad \text{with} \qquad \partial_s \tau_s^{[n]} = (\varepsilon X_1 + \ldots + \varepsilon^n X_n) \circ \tau_s^{[n]}$$

➡ Akin to post-correction *stroboscopic* averaging.

3. Hori's method

$$\tau^{[n]} = \tau_s^{[n]}\Big|_{s=\varepsilon} \qquad \text{with} \qquad \partial_s \tau_s^{[n]} = (sY_1 + \ldots + s^n Y_n) \circ \tau_s^{[n]}$$

# Summary

Starting from an ansatz

$$u^\varepsilon(t) = \Phi^\varepsilon_{t/\varepsilon} \circ \Psi^\varepsilon_t \circ \left(\Phi^{[n]}_0\right)^{-1}(u_0)$$

with $\Phi^\varepsilon$ periodic and $\Psi^\varepsilon$ a non-stiff flow, we derived

- a closed form homological equation
- a framework which extends fairly naturally to other contexts
- *modified problems* solvable with uniform accuracy
- and which naturally preserve some geometric properties

However, some obvious limitations remain:

- the geometric properties are asymptotic
- overcoming this asymptote requires *ad-hoc* tweaking
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**Thank you for you attention!**